A generalized Lagrange equation in implicit form for non-conservative mechanics

1997 J. Phys. A: Math. Gen. 30 1575

(http://iopscience.iop.org/0305-4470/30/5/021)

J. Phys. A: Math. Gen. **30** (1997) 1575–1590. Printed in the UK PII: S0305-4470(97)75714-7

A generalized Lagrange equation in implicit form for non-conservative mechanics

F Barone†, R Grassini† and G Mendella‡

† Dipartimento di Matematica e Applicazioni R Caccioppoli, Universitá di Napoli Federico II, Via Cintia, Monte S Angelo, 80126 Naples, Italy

‡ Dipartimento di Scienze Fisiche, Mostra d'Oltremare, pad. 19, 80125 Naples, Italy, and Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Italy

Received 24 June 1996, in final form 21 October 1996

Abstract. Different geometric formulations are obtained for a generalized Lagrange equation non-reducible to normal form and encompassing non-conservative dynamics.

1. Introduction

Implicit differential equations arise quite naturally in the geometrical setting of conservative mechanics, since equations of motion are deduced from variational principles and, as such, they do not exhibit the explicit (or normal) form of vector fields on some carrier space [2].

On the other hand, the vector field approach is always adopted when dealing geometrically with non-conservative mechanics [16, 1, 6, 3].

As a matter of fact, differential equations are all implicit in principle, and in no way is their basic physical meaning related to their being reducible or non-reducible to explicit form.

So conceptual clarity would require a unified implicit formulation of conservative and non-conservative mechanics.

According to Tulczyjew [17–19, 20, 22, 14], the dynamics of a conservative mechanical system, described by a Lagrangian L on the tangent bundle TQ of a configuration space Q, is governed by the implicit differential equation on cotangent bundle T^*Q generated by dL, i.e. the submanifold of TT^*Q obtained from $Im(dL) \subset T^*TQ$ through the canonical diffeomorphism of T^*TQ onto TT^*Q.

In a previous paper [2], such an equation has been intrinsically related to the implicit Euler–Lagrange equation deduced from Hamilton's variational principle.

In the present paper, the whole theory is embodied in a geometrical treatment concerning a more general kind of submanifold of TT^*Q, which encompasses the dynamics of non-conservative mechanical systems as well.

The *generalized Lagrange equation* under consideration is a submanifold $D \subset TT^*Q$ generated by any (global or local) 1-form θ on TQ.

The crucial role in analysing D is played by the Legendre morphism associated with θ (generalizing that associated with a Lagrangian [1]), which allows us to show (section 3) that D behaves like a second-order equation, in the sense that its integral curves on T^*Q turn out to be completely determined by their own projections onto Q.

0305-4470/97/501575+16$19.50

This is the central property, which leads us to recognize (section 4) that D is actually equivalent to (i.e. has the same solution curves in Q as) a genuine second-order implicit equation E (i.e. a submanifold of the second tangent bundle of Q). The integrability algorithm [15] is applied to both equations, and their respective integrable parts and constraint subsets are related to one another.

The equation E is then shown (section 5) to fit in with the geometric framework of linearly constrained systems developed in [11–13]. Hence we obtain a regularity (or hyper-regularity) condition on θ, expressed by the Legendre morphism being a local (or global) diffeomorphism, under which E (and D) can be put in normal form.

After an intrinsic analysis of the above geometrical setting, the equation E is also given (section 6) a presymplectic formulation, generalizing the one of implicit Lagrangian dynamics [9, 10, 23, 2].

The latter, extended in such a way as to include non-conservative dynamics, is then recovered (section 7) under a suitable hypothesis on θ.

Some examples (featuring a degenerate relativistic Lagrangian coupled with an electromagnetic field, a linear Lagrangian and a generalized Rayleigh dissipation function, respectively) are given in section 8.

The coordinate expressions of the main points of the above theory are finally given in section 9.

Further developments including momentum mapping and Noether theorems, as well as an extension of our scheme leading to a unified approach to constrained mechanical systems as implicit differential equations, will be the object of forthcoming papers.

2. Preliminaries

Here is a list of the main geometric tools we shall adopt in what follows.

(i). Let M be a smooth manifold.

The tangent and cotangent bundle projections onto M will be denoted by $\tau_M : TM \to M$ and $\pi_M : T^*M \to M$, respectively.

If $\psi : M \to N$ is a smooth mapping, $T\psi : TM \to TN$ is the tangent mapping of ψ, and $\psi^* : \Lambda N \to \Lambda M$ the pull-back of the exterior algebra of M into that of N by ψ.

The Liouville 1-form on T^*M will be denoted by $\vartheta_M : T^*M \to T^*T^*M : \xi \to \vartheta_M(\xi) := \xi \circ T_\xi \pi_M$.

(ii). The basic *tangent derivations* of ΛM (see [22, 14]) are the following.

Let $i_T : \Lambda M \to \Lambda TM$ be the τ_M-derivation of degree -1 which vanishes on $\Lambda^0 M$ and acts on any $\theta \in \Lambda^1 M$ by putting, for any $x \in TM$, $(i_T\theta)(x) := i_x\theta = \langle x|\theta\rangle$ (where the inner product i_x is defined by the usual pairing $\langle\cdot|\cdot\rangle$ between vectors and forms). Hence it follows that i_T acts on any $\omega \in \Lambda^2 M$ by $(i_T\omega)(x) := i_x\omega \circ T_x\tau_M$.

From i_T one also obtains a τ_M-derivation of degree 0 given by $d_T := i_T d + d i_T$ (where d denotes the exterior derivative of both ΛM and ΛTM) and satisfying, for any $\psi : M \to N$, $d_T\psi^* = (T\psi)^* d_T$.

(iii). The key role in the *geometry of a tangent bundle* $M = TQ$ (see [4, 5, 22]) is played by the vertical lifting $\nu : TQ \times_Q TQ \to TTQ$, whose restriction ν_v to the fibre $\{v\} \times T_qQ = T_qQ$ over any $v \in TQ$ (with $q := \tau_Q(v)$) maps isomorphically T_qQ onto its own tangent space at v.

On the one hand, ν transforms the tangent mapping of τ_Q into the almost-tangent structure $S : TTQ \to TTQ$ defined, for any $v \in TQ$, by $S_v := S_{|T_vTQ} := \nu_v \circ T_v\tau_Q$.

On the other hand, ν transforms the identity mapping of TQ into the dilation vector field $\Delta : TQ \to TTQ$ defined, at any $v \in TQ$, by $\Delta(v) := \nu_v(v)$.

The vertical tangent bundle $V\tau_Q$, defined as the set of all vectors $x \in TTQ$ tangent to the fibres of τ_Q, is then characterized by $S(x) = 0$.

The second tangent bundle T^2Q, defined as the set of all vectors $x \in TTQ$ satisfying $T\tau_Q(x) = \tau_{TQ}(x)$, is characterized by $S(x) = \Delta\big(\tau_{TQ}(x)\big)$.

The horizontal cotangent bundle $V^0\tau_Q$, defined as the set of all covectors $\xi \in T^*TQ$ annihilating $V\tau_Q$, is characterized by $i_S\xi := \xi \circ S_{\pi_{TQ}(\xi)} = 0$.

The above adjoint operator $i_S : T^*TQ \to T^*TQ$ also defines a derivation of degree 0 of ΛTQ vanishing on Λ^0TQ, from which one obtains another derivation of degree 1 given by $d_S := i_Sd - di_S$.

Finally recall the canonical diffeomorphism $\alpha : TT^*Q \to T^*TQ$ characterized by $\pi_{TQ} \circ \alpha = T\pi_Q$ and $d_T\vartheta_Q = \alpha^*\vartheta_{TQ}$, whose inverse α^{-1} takes any $\xi \in T^*TQ$ attached at $\pi_{TQ}(\xi) =: v$ onto an image $\alpha^{-1}(\xi) \in TT^*Q$ attached at $\tau_{T^*Q}\big(\alpha^{-1}(\xi)\big) = \xi \circ \nu_v$.

3. The generalized Lagrange equation

We shall first analyse the basic elements of a technique generating an implicit differential equation D on T^*Q from any 1-form θ on TQ. The second-order-like behaviour of such an equation will then be shown.

(i). Let θ be a 1-form on TQ.

Define the *evolution operator*

$$\mathcal{E} := \alpha^{-1} \circ \theta : TQ \to TT^*Q$$

and the *Legendre morphism*

$$\mathcal{L} := \tau_{T^*Q} \circ \mathcal{E} : TQ \to T^*Q.$$

From the commutative diagram

$$\begin{tikzcd}
T^*TQ \arrow[dr, "\pi_{TQ}"] & & TT^*Q \arrow[ll, "\alpha"'] \arrow[dl, "T\pi_Q"] \arrow[rr, "\tau_{T^*Q}"] & & T^*Q \arrow[dl, "\pi_Q"] \\
TQ \arrow[u, "\theta"] \arrow[r, "id_{TQ}"'] & TQ \arrow[rr, "\tau_Q"'] & & Q &
\end{tikzcd}$$

it follows that $\mathcal{E}$ is a section of $T\pi_Q$, i.e.

$$T\pi_Q \circ \mathcal{E} = id_{TQ} \tag{1}$$

and $\mathcal{L}$ is a bundle morphism from τ_Q to π_Q, i.e.

$$\pi_Q \circ \mathcal{L} = \tau_Q. \tag{2}$$

Moreover, for any $v \in TQ$, one has

$$\mathcal{L}(v) = \tau_{T^*Q}\left(\alpha^{-1}\big(\theta(v)\big)\right) = \theta(v) \circ \nu_v$$

and then

$$\begin{aligned}(i_S\theta)(v) &= \theta(v) \circ S_v = \theta(v) \circ \nu_v \circ T_v\tau_Q = \mathcal{L}(v) \circ T_v\tau_Q = \mathcal{L}(v) \circ T_{\mathcal{L}(v)}\pi_Q \circ T_v\mathcal{L} \\ &= \vartheta_Q\big(\mathcal{L}(v)\big) \circ T_v\mathcal{L} \\ &= (\mathcal{L}^*\vartheta_Q)(v)\end{aligned}$$

i.e.

$$i_S\theta = \mathcal{L}^*\vartheta_Q. \tag{3}$$

(ii). The image of the evolution operator

$$D := Im\,\mathcal{E}$$

will be called the *generalized Lagrange equation* on T^*Q generated by θ.

Note that, if $z \in D$, i.e. $z = \mathcal{E}(v)$ for some $v \in TQ$, then, owing to (1), $v = T\pi_Q\big(\mathcal{E}(v)\big) = T\pi_Q(z)$, whence

$$D = \big\{z \in TT^*Q \mid z = \mathcal{E}\big(T\pi_Q(z)\big)\big\}. \tag{4}$$

A smooth curve k in T^*Q is an integral curve of D, if its tangent lifting $\dot{k}$ satisfies $Im\,\dot{k} \subset D$, as follows from (4):

$$\dot{k} = \mathcal{E} \circ T\pi_Q \circ \dot{k}. \tag{5}$$

A smooth curve γ in Q will be called a base integral curve of D, if $\gamma = \pi_Q \circ k$ for some integral curve k. If γ is a base integral curve, k is uniquely determined by γ, since, owing to (5), $k = \tau_{T^*Q} \circ \dot{k} = \tau_{T^*Q} \circ \mathcal{E} \circ T\pi_Q \circ \dot{k}$ and then

$$k = \mathcal{L} \circ \dot{\gamma} \tag{6}$$

i.e. k is the *Legendre lifting* of γ.

Therefore D behaves like a ‘second-order differential equation’ on Q, whose actual unknown is γ (a smooth curve in Q) and whose solutions are the base integral curves.

In view of equations (2), (5) and (6), such solutions are characterized by the following proposition.

Proposition 1. γ is a base integral curve of D, iff

$$Im(\mathcal{L} \circ \dot{\gamma})^{\cdot} \subset D$$

i.e.

$$T\mathcal{L} \circ \ddot{\gamma} = \mathcal{E} \circ \dot{\gamma}.$$

4. Second-order formulation

A genuine second-order implicit equation E, equivalent to D, will now be worked out. The integrability algorithm (see [15]) will then be applied to both E and D, and the respective results related to one another.

(i). Let

$$E := T^2Q \cap T\mathcal{L}^{-1}(D)$$

where $E \subset T^2Q$ is a second-order differential equation on Q.

Owing to equations (2) and (4), a smooth curve c in TQ is an integral curve of E iff

$$T\tau_Q \circ \dot{c} = \tau_{TQ} \circ \dot{c} \tag{7a}$$

$$T\mathcal{L} \circ \dot{c} = \mathcal{E} \circ T\tau_Q \circ \dot{c}. \tag{7b}$$

An integral curve c is uniquely determined by the corresponding base integral curve $\gamma := \tau_Q \circ c$ in Q, since, owing to (7*a*),

$$c = \dot{\gamma}. \tag{8}$$

Solutions to E are the base integral curves, characterized (in view of (7) and (8)) by the following proposition.

Proposition 2. γ is a base integral curve of E, iff

$$Im\,\ddot{\gamma} \subset E$$

i.e.

$$T\mathcal{L} \circ \ddot{\gamma} = \mathcal{E} \circ \dot{\gamma}.$$

Propositions 1 and 2 show that equations D and E are equivalent to each other, in the sense that:

Proposition 3. D and E have the same base integral curves.

E will then be called the generalized Lagrange equation on TQ generated by θ.

(ii). E is said to be integrable at a point $x \in E$, if there exists an integral curve c of E s.t. $x \in Im\,\dot{c}$. The set $E^{(i)} \subset E$ of such points is the integrable part of E and $C^{(i)} := \tau_{TQ}(E^{(i)})$ is the motion subset of E.

Now consider the primary constraint subset $C_1 := \tau_{TQ}(E)$ and the equation $E_1 := E \cap TC_1$ (where TC_1 denotes the set of all vectors tangent to smooth curves living in C_1). E_1 is equivalent to E, i.e. E_1 has the same integral curves as E, and then $E_1^{(i)} = E^{(i)}$.

Next consider the secondary constraint subset $C_2 := \tau_{TQ}(E_1)$ and the equation $E_2 := E_1 \cap TC_2 = E \cap TC_2$. Again, E_2 is equivalent to E_1, whence $E_2^{(i)} = E_1^{(i)} = E^{(i)}$, and so on.

Let $\{C_h\}$ and $\{E_h\}$ be the sequences of constraints and equations extracted from E through the above integrability algorithm.

If, for a value f of the index, $C_f = C_{f+1}$, then f is the final step, for one has $E_f = E_h$ and $C_f = C_h$ for all $h > f$.

If E_f is integrable, i.e. $E_f = E_f^{(i)}$, one obtains $E_f = E^{(i)}$ and $C_f = C^{(i)}$.

Let $\{B_h\}$ and $\{D_h\}$ be the sequences of constraints and equations likewise extracted from D through the integrability algorithm.

As $\mathcal{L}\big(\tau_{TQ}(E)\big) = \tau_{T^*Q}\big(T\mathcal{L}(E)\big)$, from

$$T\mathcal{L}(E) \subset D$$

one obtains

$$\mathcal{L}(C_1) \subset B_1.$$

As a consequence, $T\mathcal{L}(TC_1) \subset TB_1$ and then

$$T\mathcal{L}(E_1) \subset D_1$$

whence

$$\mathcal{L}(C_2) \subset B_2$$

and so on.

As to $(E^{(i)}, C^{(i)})$ and $(D^{(i)}, B^{(i)})$, we first point out that $\mathcal{L}$ bijectively relates the integral curves of E to those of D (in view of proposition 3); hence, as the tangent liftings of integral curves sweep the whole integrable parts of the equations, we infer that $T\mathcal{L}$ maps $E^{(i)}$ onto $D^{(i)}$, and then $\mathcal{L}$ maps $C^{(i)}$ onto $B^{(i)}$.

In conclusion, we have the following proposition.

Proposition 4. At each step of the integrability algorithm, one has

$$T\mathcal{L}(E_h) \subset D_h \qquad \mathcal{L}(C_h) \subset B_h$$

whereas

$$T\mathcal{L}(E^{(i)}) = D^{(i)} \qquad \mathcal{L}(C^{(i)}) = B^{(i)}.$$

5. Linearly constrained formulation

The equation E will be shown to fit in with the geometrical framework of linearly constrained systems (see [11–13]). Regularity conditions on θ under which E (and D) can be put in normal form, will thereby be obtained.

(i). Note that, for any $x \in E$, one has $T\mathcal{L}(x) \in D$ – i.e. $T\mathcal{L}(x) = \mathcal{E} \circ T\pi_Q \circ T\mathcal{L}(x) = \mathcal{E} \circ T\tau_Q(x)$ – and $T\tau_Q(x) = \tau_{TQ}(x)$, whence

$$T\mathcal{L}(x) = \mathcal{E} \circ \tau_{TQ}(x). \tag{9}$$

Conversely, for any $x \in TTQ$ satisfying (9), one has $T\mathcal{L}(x) \in D$ and $T\tau_Q(x) = T\pi_Q \circ T\mathcal{L}(x) = T\pi_Q \circ \mathcal{E} \circ \tau_{TQ}(x) = \tau_{TQ}(x)$, i.e. $x \in E$.

So we obtain the following proposition.

Proposition 5. $E = \{x \in TTQ \mid T\mathcal{L}(x) = \mathcal{E} \circ \tau_{TQ}(x)\}$.

The algebraic equation (9) also reads

$$A(x) = \sigma \circ \tau_{TQ}(x)$$

where $A := (\tau_{TQ}, T\mathcal{L}) : TTQ \to TQ \times_{T^*Q} TT^*Q$ is a vector bundle morphism from τ_{TQ} to $\rho_{TQ} := \mathcal{L}^*(\tau_{T^*Q})$ and $\sigma := (id_{TQ}, \mathcal{E}) : TQ \to TQ \times_{T^*Q} TT^*Q$ is a section of ρ_{TQ}.

Proposition 5 then shows that E is the differential equation on TQ defined by the linearly constrained system $(\tau_{TQ}, \rho_{TQ}, A, \sigma)$.

(ii). For any $v \in TQ$, the set of solutions to the linear equation $T_v\mathcal{L}(x) = \mathcal{E}(v)$ (if non-empty) is an affine subspace of T_vTQ modelled on $ker(T_v\mathcal{L})$; it then reduces to a singleton $\{x_v\} \subset T^2Q$ iff $T_v\mathcal{L}$ is injective.

So, if $\mathcal{L}$ is a local diffeomorphism, and only in that case, E is reducible to normal form $E = Im\,X$, with $X : v \in TQ \to x_v \in T^2Q$ SODE vector field on TQ.

θ will be said to be a *regular* 1-form when $\mathcal{L}$ is a local diffeomorphism, and then:

Proposition 6. E is reducible to normal form, iff θ is regular.

θ will be said to be a *hyper-regular* 1-form when $\mathcal{L}$ is a diffeomorphism.

In that case, we have $E = Im\,X$ and we can also define $Z := \mathcal{L}_*X$ (the push-forward of X by $\mathcal{L}$) by $Z = T\mathcal{L} \circ X \circ \mathcal{L}^{-1}$.

On the one hand, we have $Im\,Z \subset D$.

On the other hand, for any $z \in D$, from

$$p := \tau_{T^*Q}(z)$$

we obtain

$$\tau_{T^*Q}\big(Z(p)\big) = \tau_{T^*Q}(z)$$

$$\tau_{T^*Q} \circ \mathcal{E} \circ T\pi_Q \circ Z(p) = \tau_{T^*Q} \circ \mathcal{E} \circ T\pi_Q(z)$$

$$\mathcal{L} \circ T\pi_Q \circ Z(p) = \mathcal{L} \circ T\pi_Q(z)$$

$$T\pi_Q \circ Z(p) = T\pi_Q(z)$$

$$\mathcal{E} \circ T\pi_Q \circ Z(p) = \mathcal{E} \circ T\pi_Q(z)$$

$$Z(p) = z$$

i.e. $D \subset Im\, Z$.

So we obtain the following proposition.

Proposition 7. If θ is hyper-regular, E and D are both reducible to normal form:

$$E = Im\, X \qquad D = Im\, Z$$

with

$$Z = \mathcal{L}_* X.$$

6. Presymplectic formulation

Further geometrical objects (a presymplectic 2-form and an 'energy' 1-form on TQ associated with θ) will emerge from an intrinsic analysis of the above formulation of the equation E. Once expressed in terms of such objects, E will exhibit a presymplectic formulation generalizing that of implicit Lagrangian dynamics.

(i). In view of section 5(i), one has that $x \in E$ iff

$$x \in T^2 Q$$

and

$$\alpha\big(T\mathcal{L}(x)\big) - \theta\big(\tau(x)\big) = 0$$

where

$$\tau := \tau_{TQ} \circ \iota$$

and

$$\iota : T^2 Q \hookrightarrow TTQ.$$

(ii). Now note that, for any $x \in T^2 Q$,

$$T_x\tau = T_x(\tau_{TQ} \circ \iota) = T_x(T\tau_Q \circ \iota) = T_x(T\pi_Q \circ T\mathcal{L} \circ \iota) = T_x(\pi_{TQ} \circ \alpha \circ T\mathcal{L} \circ \iota)$$

$$= T_{\alpha(T\mathcal{L}(x))}\pi_{TQ} \circ T_x(\alpha \circ T\mathcal{L} \circ \iota)$$

and then

$$\alpha\big(T\mathcal{L}(x)\big) \circ T_x\tau = \vartheta_{TQ}\left(\alpha\big(T\mathcal{L}(x)\big)\right) \circ T_x(\alpha \circ T\mathcal{L} \circ \iota)$$

$$= \big((\alpha \circ T\mathcal{L} \circ \iota)^* \vartheta_{TQ}\big)(x).$$

Moreover, owing to (3),

$$(\alpha \circ T\mathcal{L} \circ \iota)^*\vartheta_{TQ} = \iota^* T\mathcal{L}^*\alpha^*\vartheta_{TQ} = \iota^* T\mathcal{L}^* d_T\vartheta_Q = \iota^* d_T\mathcal{L}^*\vartheta_Q = \iota^* d_T i_S\theta$$
$$= \iota^* d i_T i_S\theta + \iota^* i_T d i_S\theta.$$

On the other hand

$$(\iota^* i_T i_S\theta)(x) = (i_T i_S\theta)(x) = \langle x \mid i_S\theta\rangle = \langle S(x) \mid \theta\rangle = \langle \Delta\big(\tau(x)\big) \mid \theta\rangle = \big(i_\Delta\theta\big)\big(\tau(x)\big)$$
$$= \big(\tau^* i_\Delta\theta\big)(x)$$

i.e.

$$\iota^* i_T i_S\theta = \tau^* i_\Delta\theta.$$

Hence

$$\big(\alpha\big(T\mathcal{L}(x)\big) - \theta\big(\tau(x)\big)\big) \circ T_x\tau = \big(\tau^* d i_\Delta\theta + \iota^* i_T d i_S\theta - \tau^*\theta\big)(x).$$

If we introduce the *presymplectic 2-form*

$$\omega := -d i_S\theta \tag{10}$$

(which need not be of constant rank and, owing to (3), is symplectic iff θ is regular) and the *energy 1-form*

$$\eta := d i_\Delta\theta - \theta \tag{11}$$

the above result reads

$$\big(\alpha\big(T\mathcal{L}(x)\big) - \theta\big(\tau(x)\big)\big) \circ T_x\tau = \big(\tau^*\eta - \iota^* i_T\omega\big)(x) = \eta\big(\tau(x)\big) \circ T_x\tau - \big(i_T\omega\big)(x) \circ T_x\iota$$
$$= \eta\big(\tau(x)\big) \circ T_x\tau - i_x\omega \circ T_x\tau_{TQ} \circ T_x\iota$$
$$= \big(\eta\big(\tau(x)\big) - i_x\omega\big) \circ T_x\tau$$

whence ($T_x\tau$ being surjective)

$$\alpha\big(T\mathcal{L}(x)\big) - \theta\big(\tau(x)\big) = \eta\big(\tau(x)\big) - i_x\omega.$$

(iii). From (i) and (ii), we obtain the following proposition.

Proposition 8. $E = \big\{x \in T^2Q \mid i_x\omega = \eta\big(\tau(x)\big)\big\}$.

7. Non-conservative Lagrangian formulation

A special assumption on θ will introduce a Lagrangian formalism in the presymplectic setting of the equation E. Implicit Lagrangian dynamics, extended in such a way as to include non-conservative systems, will thereby be obtained.

(i). Let us assume $i_S\theta$ to be a d_S-exact 1-form, i.e.

$$i_S\theta = d_S L \tag{12}$$

for some smooth *Lagrangian* function L on TQ.

This amounts to saying that $\theta = dL + F$ with $i_S F = 0$.

Note that the above splitting of θ into the sum of an exact 1-form dL and a horizontal 1-form F on TQ, is determined up to a gauge choice given by

$$(L, F) \mapsto (L - \tau_Q^* V, F + \tau_Q^* dV)$$

V being an arbitrary smooth function on Q. As a consequence, when we refer to a gauge (L, F), 1-form F, if non-null, will be assumed to be non-exact.

(ii). With reference to a gauge (L, F), the equation E can be formulated as follows.

Owing to (10), and recalling that F is horizontal, one has

$$\begin{aligned}\omega &= -di_S dL - di_S F\\ &= \omega_L\end{aligned}$$

with $\omega_L := -dd_S L$ (Poincaré–Cartan 2-form).

Owing to (11), and recalling that Δ is vertical, one has

$$\begin{aligned}\eta &= di_\Delta dL + di_\Delta F - dL - F\\ &= dE_L - F\end{aligned}$$

with $E_L := \Delta L - L$ (*energy* function).

Then put

$$[L] : T^2Q \to V^0\tau_Q : x \mapsto dE_L\big(\tau(x)\big) - i_x\omega_L$$

(Euler–Lagrange morphism).

From proposition 8, it follows that:

Proposition 9.

$$\begin{aligned}E &= \big\{x \in T^2Q \mid i_x\omega_L = dE_L\big(\tau(x)\big) - F\big(\tau(x)\big)\big\}\\ &= \big\{x \in T^2Q \mid [L](x) = F\big(\tau(x)\big)\big\}.\end{aligned}$$

The base integral curves of E are then characterized by

$$[L] \circ \ddot{\gamma} = F \circ \dot{\gamma} \tag{13}$$

which is the equation of motion of a mechanical system, described by a Lagrangian L and acted upon by an *external force field* F.

According to (13), the motions of the system are simply conceived as those which deviate from the *comparison* or *inertial* motions, characterized by Euler–Lagrange equation $[L] \circ \ddot{\gamma} = 0$ (see [2]), in that their *inertial force* $-[L] \circ \ddot{\gamma}$ is balanced by the external force $F \circ \dot{\gamma}$.

Note that any other admissible gauge would lead to different specifications of the (conventional) notions of inertia and force, without of course altering the (observable) class of motions.

(iii). With reference to a gauge (L, F), as well as the energy E_L of L, one can define the *power* Π_F of F by putting

$$\Pi_F : TQ \to \mathbb{R} : v \mapsto \Pi_F(v) := \langle v \mid \tilde{F}(v)\rangle$$

where $\tilde{F} : TQ \to T^*Q$ is the bundle morphism characterized by $F = \tilde{F}^*\vartheta_Q$, i.e. for any $v \in TQ$, $F(v) = \tilde{F}(v) \circ T_v\tau_Q$.

From proposition 9, one then infers the following *energy balance law*

$$\langle x \mid dE_L\rangle = \Pi_F\big(\tau(x)\big) \qquad \forall\, x \in E.$$

Along each base integral curve, the energy balance law reads

$$\frac{d}{dt}(E_L \circ \dot{\gamma}) = \Pi_F \circ \dot{\gamma}.$$

If $\Pi_F = 0$ or $\Pi_F \leqslant 0$, conservation or dissipation of energy E_L along the motions will follow.

Note that any other admissible gauge (L', F') would lead to $E_{L'} = E_L + \tau_Q^* V$ and then the above conservation or dissipation law would be concerning the total energy obtained by adding up the energy $E_{L'}$ of L' and the potential energy $-\tau_Q^* V$ of $F' - F$.

8. Examples

Applications to relativistic dynamics, linear Lagrangians and Rayleigh dissipation functions now follow.

(i). Let $K : TQ \to \mathbb{R} \ : \ v \mapsto \frac{1}{2}\langle v, v\rangle$ be the kinetic energy associated with a Lorentz metric $\langle\cdot,\cdot\rangle$ of index $\dim Q - 1$.

On the time-like open subset $C := \{v \in TQ \,|\, K(v) > 0\}$, consider a 1-form θ of type (12), admitting a gauge (L, F) where

$$L := m\sqrt{2K}$$

(with $m > 0$) is a 'relativistic' Lagrangian (see [21]) and

$$F := \frac{1}{\sqrt{2K}}\Phi$$

is defined by an 'electromagnetic' force field

$$\Phi := e\, i_T \mathbf{F}$$

(with $e \in \mathbb{R}$ and $\mathbf{F} \in \Lambda_2 Q$).

We remark that

$$\omega_L = -\frac{m}{\sqrt{2K}} dd_S K - d\left(\frac{m}{\sqrt{2K}}\right) \wedge d_S K = \frac{m}{\sqrt{2K}}\left(\omega_K + \frac{1}{2K} dK \wedge d_S K\right)$$

$$= \frac{m}{\sqrt{2K}}\left(\omega_K - \frac{1}{2K} dK \wedge i_\Delta \omega_K\right)$$

and

$$E_L = \frac{m}{\sqrt{2K}} \Delta K - m\sqrt{2K} = 0.$$

Moreover recall that, for all $v \in TQ$,

$$\Phi(v) = e\, i_v \mathbf{F} \circ T_v \tau_Q$$

and then

$$\tilde{\Phi}(v) = e\, i_v \mathbf{F}$$

whence

$$\Pi_\Phi(v) = 0.$$

Now, for any $x \in T^2 Q$ (with $v := \tau(x) \in C$), one has $x \in E$, i.e.

$$i_x \omega_L = dE_L(v) - F(v)$$

iff

$$i_x \omega_K = i_{\Gamma(v)} \omega_K + g(x) i_{\Delta(v)} \omega_K$$

i.e.

$$x = \Gamma(v) + g(x)\Delta(v)$$

where Γ is the SODE vector field determined by

$$i_\Gamma \omega_K = dK - \frac{1}{m}\Phi$$

and

$$g := \frac{1}{2K \circ \tau} d_T K.$$

The above condition on x amounts to saying that

$$x = \Gamma(v) + a\Delta(v)$$

for some $a \in \mathbb{R}$, since

$$\begin{aligned} g(x) = \frac{1}{2K(v)}\langle x \mid dK(v)\rangle &= \frac{1}{2K(v)}\langle \Gamma(v) \mid dK(v)\rangle + \frac{a}{2K(v)}\langle \Delta(v) \mid dK(v)\rangle \\ &= \frac{1}{2mK(v)}\langle \Gamma(v) \mid \Phi(v)\rangle + a = \frac{1}{2mK(v)}\langle T\tau_Q\big(\Gamma(v)\big) \mid \tilde{\Phi}(v)\rangle + a \\ &= \frac{1}{2mK(v)}\Pi_\Phi(v) + a \\ &= a. \end{aligned}$$

So we obtain

$$E = \left\{x \in T^2Q \mid \tau(x) \in C\,,\ x = \Gamma\big(\tau(x)\big) + a\Delta\big(\tau(x)\big) \quad (a \in \mathbb{R})\right\}\ .$$

Let γ be a base integral curve of E, i.e.

$$\ddot{\gamma} = \Gamma \circ \dot{\gamma} + a(\Delta \circ \dot{\gamma})$$

(a being a real-valued function defined on the domain of γ).

Along γ, one has

$$\begin{aligned} \frac{d}{dt}(K \circ \dot{\gamma}) = \langle \ddot{\gamma} \mid dK \circ \dot{\gamma}\rangle &= \langle \Gamma \mid dK\rangle \circ \dot{\gamma} + a\langle \Delta \mid dK\rangle \circ \dot{\gamma} \\ &= a(2K \circ \dot{\gamma}). \end{aligned}$$

Hence it follows that γ obeys the constraint

$$K \circ \dot{\gamma} = 1$$

iff it is a base integral curve of Γ (i.e. $a = 0$) starting from initial conditions belonging to $K^{-1}(1)$.

If Q is the space-time manifold of general relativity, any such curve is a possible world line (parametrized by proper time) of a test particle with rest mass m and electric charge e, moving in a gravitational field K and acted upon by an electromagnetic force field Φ.

(ii). Let θ be a 1-form on TQ of type (12), admitting a gauge (L, F) with

$$L = i_T\lambda$$

λ being a 1-form on Q (linear Lagrangian).

As

$$\omega_L = -\tau_Q^* d\lambda$$

and

$$E_L = 0$$

for any $x \in T^2Q$, putting $v := \tau(x)$, one has

$$[L](x) = i_x \tau_Q^* d\lambda = i_v d\lambda \circ T_v \tau_Q$$

and then

$$[L](x) - F\big(\tau(x)\big) = \big(i_v d\lambda - \tilde{F}(v)\big) \circ T_v \tau_Q.$$

Hence

$$E = \tau^{-1}(C)$$

with

$$C := \big\{v \in TQ \mid i_v d\lambda = \tilde{F}(v)\big\}.$$

Actually E reduces to a first-order equation on Q, namely its final constraint C, since (for any smooth curve γ in Q)

$$Im\, \ddot{\gamma} \subset E \qquad \text{iff} \quad Im\, \dot{\gamma} \subset C.$$

If $\lambda = 0$ (i.e. $L = 0$ up to gauge transformations) and $\tilde{F} = \phi \circ \tau_Q$ (with $\phi \in \Lambda^1 Q$), one has

$$C = \tau_Q^{-1}(W)$$

with

$$W := \big\{q \in Q \mid \phi(q) = 0\big\}.$$

In that case, C in turn reduces to a holonomic constraint, namely W, since

$$Im\, \dot{\gamma} \subset C \qquad \text{iff} \quad Im\, \gamma \subset W.$$

(iii). Let θ be a 1-form of type (12), admitting a gauge (L, F) with

$$F = -d_S \mathfrak{F}$$

$\mathfrak{F}$ being a real-valued smooth function on TQ.

For any $v \in TQ$, one has $F(v) = -d\mathfrak{F}(v) \circ S_v$ or, equivalently, $\tilde{F}(v) = -d\mathfrak{F}(v) \circ \nu_v$ whence $\langle v \mid \tilde{F}(v)\rangle = -\langle \Delta(v) \mid d\mathfrak{F}(v)\rangle$, i.e.

$$\Pi_F = -\Delta \mathfrak{F}.$$

As a consequence, an energy dissipation law holds along the motions if $\Delta\mathfrak{F} \geqslant 0$.

That is the case, e.g., when $\mathfrak{F}$ is a Rayleigh dissipation function, i.e. the quadratic form of a positive-semidefinite, symmetric, $(0, 2)$ tensor field k on Q (such a function, on a Riemmanian manifold $(Q, \langle\cdot,\cdot\rangle)$, corresponds to a frictional force, since, regarding k as a non-negative self-adjoint vector 1-form on Q, one gets $\mathfrak{F}(v) = \frac{1}{2}\langle k(q) \cdot v\,,\, v\rangle$ for any $v \in T_q Q$, and then $\tilde{F}(v) = -\langle k(q) \cdot v)$.

In such a case, one obtains the classical dissipation condition (see [8])

$$\Pi_F = -2\mathfrak{F} \leqslant 0.$$

9. Coordinate expression

It is instructive to follow the construction described from sections 3 to 7 in a local chart of Q (and corresponding charts of the relevant tangent and cotangent bundles). Our coordinate notation will omit indices and will then read as standard matrix notation.

(i). Recall that (see [22])

$$\alpha^{-1} : \big(q, v/r, s\big) \in T^*TQ \mapsto \big(q, s/v, r\big) \in TT^*Q.$$

Hence

$$\begin{tikzcd}
\mathcal{E} := \alpha^{-1} \circ \theta : \big(q, v\big) \in TQ \arrow[r, "\theta"] & \big(q, v/\theta_q(q, v), \theta_v(q, v)\big) \in T^*TQ \arrow[d, "\alpha^{-1}"] \\
 & \big(q, \theta_v(q, v)/v, \theta_q(q, v)\big) \in TT^*Q
\end{tikzcd}$$

and

$$\mathcal{L} := \tau_{T^*Q} \circ \mathcal{E} : \big(q, v\big) \in TQ \mapsto \big(q, \theta_v(q, v)\big) \in T^*Q.$$

For any

$$z \equiv (q, p/\dot{q}, \dot{p}) \in TT^*Q$$

one has $T\pi_Q(z) \equiv (q, \dot{q}) \in TQ$ and then

$$\mathcal{E} \circ T\pi_Q(z) \equiv \big(q, \theta_v(q, \dot{q})/\dot{q}, \theta_q(q, \dot{q})\big) \in TT^*Q.$$

So $z \in D := Im\,\mathcal{E}$, i.e. $z = \mathcal{E} \circ T\pi_Q(z)$, iff the coordinates $(q, p/\dot{q}, \dot{p})$ satisfy

$$p = \theta_v(q, \dot{q}) \qquad \dot{p} = \theta_q(q, \dot{q}). \tag{14}$$

D is then the submanifold of TT^*Q locally described by equations (14).

Now let $k \equiv (p, q)$ (with $q = q(t)$, $p = p(t)$) be a smooth curve in the given coordinate domain on T^*Q, and $\dot{k} \equiv (q, p/\dot{q}, \dot{p})$ (with $\dot{q} = dq/dt$, $\dot{p} = dp/dt$) its tangent lifting.

From the above description of D, it follows that k is an integral curve of D, i.e. $Im\,\dot{k} \subset D$, iff the functions $\big(q(t), p(t)\big)$ satisfy the first-order implicit differential equations (14).

As a consequence, projection $\gamma := \pi_Q \circ k$ will be represented by functions $q(t)$ satisfying the second-order implicit differential equations

$$\frac{d}{dt}\,\theta_v(q, \dot{q}) = \theta_q(q, \dot{q}) \tag{15}$$

which then locally characterize the base integral curves of D.

We remark that equations (14) and (15) locally confirm that the Legendre lifting $\gamma \mapsto \mathcal{L} \circ \dot{\gamma}$ maps base integral curves onto integral curves and, as a mapping between such classes of curves, is invertible, its two-sided inverse being the projection $k \mapsto \pi_Q \circ k$.

(ii). For any

$$x \equiv (q, v/\dot{q}, \dot{v}) \in TTQ$$

one has

$$T\mathcal{L}(x) \equiv \left(q, \theta_v(q, v)/\dot{q}, \frac{\partial\theta_v}{\partial q}\,\dot{q} + \frac{\partial\theta_v}{\partial v}\,\dot{v}\right) \in TT^*Q$$

(where the partial derivatives are evaluated at (q, v)), and

$$\mathcal{E} \circ \tau_{TQ}(x) = \big(q, \theta_v(q, v)/v, \theta_q(q, v)\big) \in TT^*Q.$$

So $x \in E := T^2Q \cap T\mathcal{L}^{-1}(D)$, i.e. $T\mathcal{L}(x) = \mathcal{E} \circ \tau_{TQ}(x)$, iff the coordinates $(q, v/\dot{q}, \dot{v})$ satisfy

$$\dot{q} = v \tag{16a}$$

$$\frac{\partial\theta_v}{\partial q}\,\dot{q} + \frac{\partial\theta_v}{\partial v}\,\dot{v} = \theta_q(q, v). \tag{16b}$$

E is then the submanifold of TTQ (T^2Q) locally described by equations (16) (equation (16b)).

Now let $c \equiv (q, v)$ (with $q = q(t),\ v = v(t)$) be a smooth curve in the given coordinate domain on TQ, and $\dot{c} = (q, v/\dot{q}, \dot{v})$ (with $\dot{q} = dq/dt,\ \dot{v} = dv/dt$) its tangent lifting.

From the above description of E, it follows that c is an integral curve of E, i.e. $Im\,\dot{c} \subset E$, iff the functions $\big(q(t), v(t)\big)$ satisfy the first-order implicit differential equations (16).

As a consequence, the projection $\gamma := \tau_Q \circ c$ will be represented by functions $q(t)$ satisfying equations (15), which then also locally characterize the base integral curves of E.

We remark that equations (15) and (16) locally confirm that the tangent lifting $\gamma \mapsto \dot{\gamma}$ maps base integral curves onto integral curves and, as a mapping between such classes of curves, is invertible, its two-sided inverse being the projection $c \mapsto \tau_Q \circ c$.

The same local description, of course, will be obtained from the coordinate expression of the presymplectic formalism.

Standard computations show that $\omega := -di_S\theta$ has a block-matrix of components given by

$$\begin{bmatrix} \dfrac{\partial\theta_v}{\partial q} - \left(\dfrac{\partial\theta_v}{\partial q}\right)^T & \dfrac{\partial\theta_v}{\partial v} \\ -\left(\dfrac{\partial\theta_v}{\partial v}\right)^T & 0 \end{bmatrix}.$$

As a consequence, for any $x \equiv (q, v/\dot{q}, \dot{v}) \in TTQ$, one has

$$(i_x\omega)_q = \left(\frac{\partial\theta_v}{\partial q}\right)^T \dot{q} - \frac{\partial\theta_v}{\partial q}\dot{q} - \frac{\partial\theta_v}{\partial v}\dot{v}$$

$$(i_x\omega)_v = \left(\frac{\partial\theta_v}{\partial v}\right)^T \dot{q}.$$

Moreover, from $\eta := di_\Delta\theta - \theta$, one obtains

$$\big(\eta\big(\tau(x)\big)\big)_q = \left(\frac{\partial\theta_v}{\partial q}\right)^T v - \theta_q(q, v)$$

$$\big(\eta\big(\tau(x)\big)\big)_v = \left(\frac{\partial\theta_v}{\partial v}\right)^T v.$$

Hence

$$\big(\eta\big(\tau(x)\big) - i_x\omega\big)_q = \left(\frac{\partial\theta_v}{\partial q}\right)^T (v - \dot{q}) + \frac{\partial\theta_v}{\partial q}\dot{q} + \frac{\partial\theta_v}{\partial v}\dot{v} - \theta_q(q, v)$$

$$\big(\eta\big(\tau(x)\big) - i_x\omega\big)_v = \left(\frac{\partial\theta_v}{\partial v}\right)^T (v - \dot{q}).$$

So $x \in E$, i.e. $x \in T^2Q$ and $\eta\big(\tau(x)\big) - i_x\omega = 0$, iff the coordinates $(q, v/\dot{q}, \dot{v})$ satisfy equations (16).

Note that, if γ is a smooth curve in Q represented by functions $q = q(t)$, then $\eta\circ\dot{\gamma} - i_{\ddot{\gamma}}\omega$ is a section of $V^0\tau_Q$ along $\dot{\gamma}$ admitting components given by $\big(d\theta_v(q,\dot{q})/dt - \theta_q(q,\dot{q}), 0\big)$. As a consequence, we reobtain that γ is a base integral curve of E (i.e. $Im\,\ddot{\gamma} \subset E$ or, equivalently, $\eta\circ\dot{\gamma} - i_{\ddot{\gamma}}\omega = 0$) iff the functions $q(t)$ satisfy equations (15).

(iii). If $\theta = dL + F$ with F horizontal (and then $F_v = 0$), one has $\theta_q = \partial L/\partial q + F_q$ and $\theta_v = \partial L/\partial v$.

Equations (14), (16) and (15) then read

$$p = \frac{\partial L}{\partial v} \qquad \dot{p} = \frac{\partial L}{\partial q} + F_q$$

$$\dot{q} = v \qquad \left[\frac{\partial}{\partial q}\left(\frac{\partial L}{\partial v}\right)\right]\dot{q} + \left[\frac{\partial}{\partial v}\left(\frac{\partial L}{\partial v}\right)\right]\dot{v} - \frac{\partial L}{\partial q} = F_q(q,v)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_q(q,\dot{q})$$

which are the familiar coordinate Lagrange equations meant as local implicit differential equations on T^*Q, TQ and Q, respectively.

Acknowledgment

We would like to thank Professor G Marmo for reading the manuscript and for stimulating discussions.

References

[1] Abraham R and Marsden J E 1978 *Foundations of Mechanics* (Reading, MA: Benjamin)

[2] Barone F and Grassini R 1997 On the second-order Euler–Lagrange equation in implicit form *Ric. Mat.* to appear

[3] Cantrijn F 1984 Symplectic approach to nonconservative mechanics *J. Math. Phys.* **25** 271–6

[4] Crampin M 1983 Tangent bundle geometry for Lagrangian dynamics *J. Phys. A: Math. Gen.* **16** 3755–72

[5] de Léon M and Rodrigues P R 1989 *Methods of Differential Geometry in Analytical Mechanics* (Amsterdam: North-Holland)

[6] de Ritis R, Marmo G, Platania G and Scudellaro P 1983 Inverse problem in classical mechanics: dissipative systems *Int. J. Theor. Phys.* **22** 931–46

[7] Godbillon C 1969 *Géométrie Différentielle et Mécanique Analytique* (Paris: Hermann)

[8] Goldstein H 1980 *Classical Mechanics* (Reading, MA: Addison-Wesley)

[9] Gotay M J and Nester J 1979 Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem *Ann. Inst. H Poincaré* **30** 129–42

[10] Gotay M J and Nester J 1980 Presymplectic Lagrangian systems II: the second-order equation problem *Ann. Inst. H Poincaré* **32** 1–13

[11] Grácia X and Pons J M 1989 On an evolution operator connecting Lagrangian and Hamiltonian formalism *Lett. Math. Phys.* **17** 175–80

[12] Grácia X and Pons J M 1991 Constrained systems: a unified geometric approach *Int. J. Theor. Phys.* **30** 511–6

[13] Grácia X and Pons J M 1992 A generalized geometric framework for constrained systems *Diff. Geom. Appl.* **2** 223–47

[14] Marmo G, Mendella G and Tulczyjew W M 1992 Symmetries and constants of the motion for dynamics in implicit form *Ann. Inst. H Poincaré* **57** 147–66

[15] Marmo G, Mendella G and Tulczyjew W M 1995 Integrability of implicit differential equations *J. Phys. A.: Math. Gen.* **28** 149–63

[16] Shahshahani S 1972 Dissipative systems on manifolds *Invent. Math.* **16** 177–90

[17] Tulczyjew W M 1974 Hamiltonian systems, Lagrangian systems and the Legendre transformation *Symp. Math.* **16** 247–58

[18] Tulczyjew W M 1976 Le sous-varietes Lagrangiennes et la dynamique Hamiltonienne *CR Acad. Sci. Paris* **283** 15–8

[19] Tulczyjew W M 1976 Le sous-varietes Lagrangiennes et la dynamique Lagrangienne *CR Acad. Sci. Paris* **283** 675–8

[20] Tulczyjew W M 1977 The Legendre transformation *Ann. Inst. H Poincaré* **27** 101–14

[21] Tulczyjew W M 1977 A symplectic formulation of relativistic particle dynamics *Acta Phys. Pol.* B **8** 431–47

[22] Tulczyjew W M 1989 *Geometric Formulations of Physical Theories* (Naples: Bibliopolis)

[23] Woodhouse N 1980 *Geometric Quantization* (Oxford: Clarendon)